ADVANCED PROGRAMME MATHEMATICS: PAPER II

Time: 1 hour   100 marks

PLEASE READ THE FOLLOWING INSTRUCTIONS CAREFULLY

1. This question paper consists of 14 pages and an Information Booklet of 4 pages (i–iv). Please check that your question paper is complete.

2. This question paper consists of THREE modules:

   Choose **ONE** of the **THREE** modules:

   MODULE 2: STATISTICS (100 marks) OR
   MODULE 3: FINANCE AND MODELLING (100 marks) OR
   MODULE 4: MATRICES AND GRAPH THEORY (100 marks)

3. Non-programmable and non-graphical calculators may be used.

4. All necessary calculations must be clearly shown and writing must be legible.

5. Diagrams have not been drawn to scale.

6. **Rounding of final answers.**

   MODULE 2: Four decimal places, unless otherwise stated.

   MODULE 3: Two decimal places, unless otherwise stated.

   MODULE 4: Two decimal places, unless otherwise stated.

MODULE 2   STATISTICS

QUESTION 1

1.1   The probability that your birthday is on a Sunday in a randomly chosen year
        is $\frac{1}{7}$.

   (a)   Seven years are chosen randomly. Find the probability that on more
           than one of these years your birthday is on a Sunday.   (7)

   (b)   60 years are chosen randomly. Use a suitable approximation to find
           the probability that your birthday would be on a Sunday in at least 13
           of these years. Justify the use of your approximation.   (8)

1.2   Nic buys 5 novels and 2 biographies from a bookshop. He chooses 3 of these
        books at random to take with him on a week away to the bush.

   (a)   Determine the probability that he chooses 2 novels.   (7)

   (b)   Given that the random variable X represents the number of novels Nic
           chooses. Show all the possible values of X and their corresponding
           probabilities in a table.   (5)

x				
$P(X = x)$				

   (c)   Hence, calculate the variance of X.   (4)

[31]

QUESTION 2

2.1 Keshav noted that the time an adult sleeps per night follows a normal distribution with a mean of 7,5 hours and a standard deviation of 45 minutes.

    (a) Find the probability that a randomly chosen adult sleeps less than 8 hours per night. (5)

    (b) Determine the interquartile range of the distribution. (6)

    (c) If 200 adults were randomly selected from the distribution, estimate how many of these adults sleep less than 8 hours per night. (2)

2.2 Takudzwanashe did a study on how long the Coronavirus survives on the outside of a surgical mask.

- The confidence interval he calculated was based on a sample of 50 surgical masks.
- Each mask had been equally exposed to the virus at room temperature.
- The confidence interval was (5,99; 8,01) days.

Outside of surgical mask

    (a) Find the sample mean of the data. (2)

    (b) Given that the standard deviation is 3,5 days, determine the $\alpha\,\%$ confidence interval for the population mean. (8)

[23]

QUESTION 3

3.1   Multiple Choice:

(a)   Kieran and Nate both construct 95% confidence intervals for a particular population mean. Kieran uses a random sample of size 40 and Nate a random sample of size 35.

Which of the following is true?

A   Kieran's interval has a greater degree of confidence.
B   Nate's interval has a greater degree of confidence.
C   The total width of Kieran's confidence interval estimate is narrower.
D   The total width of Nate's confidence interval estimate is narrower.   (2)

(b)   The null hypothesis is best understood to be:

A   A statement of probability.
B   The same as the research hypothesis.
C   A statement about the mean.
D   A statement that there is no difference.   (2)

(c)   The level of significance of a hypothesis test indicates:

A   The chance we are right in failing to reject the null hypothesis.
B   How significant the difference is between the population mean and the sample mean.
C   The chance we are wrong in rejecting the null hypothesis.
D   Whether to reject or fail to reject the null hypothesis.   (2)

(d)   A manufacturer of tennis racquets claims that her racquets, with regular play, will last at least two years on average. Luke tests a sample and calculates the test statistic to be 1,75. Using a 5% level of significance, the conclusion would be:

A   There is sufficient evidence for the manufacturer's claim to be considered correct.
B   There is insufficient evidence for the manufacturer's claim to be considered correct.
C   There is sufficient evidence for the manufacturer's claim to be considered incorrect.
D   There is insufficient evidence for the manufacturer's claim to be considered incorrect.   (2)

3.2   A teacher informs her learners that it takes an average time of 22 minutes to complete the homework with a standard deviation of 5 minutes. Thendo claims it takes a bit longer. A hypothesis test of this claim at the 7% level of significance was carried out using a random sample of 30 homework tasks.

(a)   State the null and alternate hypothesis.   (2)

(b)   Determine the possible values of the sample mean that will result in a conclusion that rejects the null hypothesis.   (6)

[16]

QUESTION 4

Given the probability density function for the random variable X, where k is a positive constant.

$$f(x) = \begin{cases} \dfrac{2}{k^2}x & 0 < x < k \\ 0 & \text{otherwise} \end{cases}$$
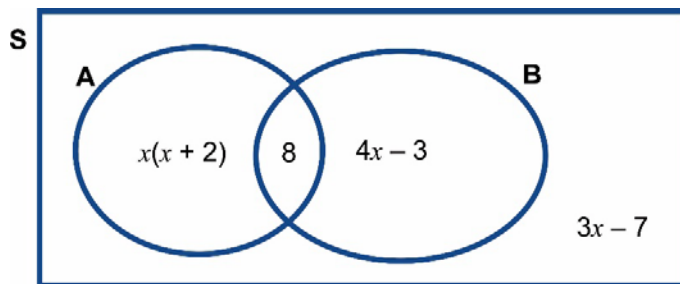
(a) Verify that f is a density function for all values of k. (6)

(b) Given that the lower quartile is 6, determine the value for k. (6)

[12]

QUESTION 5

5.1 The Venn diagram given below has a sample size of 50.

(a) Determine the value of x. (4)

(b) Hence, calculate $P(A|B')$. (3)

(c) Are events A and B independent? Justify your answer mathematically. (3)

5.2 Five letters are selected at random from the nine letters of the word INFLUENCE.

Find the number of different selections if the 5 letters include at least one N and at most one E. (8)

[18]

Total for Module 2: 100 marks

MODULE 3   FINANCE AND MODELLING

QUESTION 1

ABC taxis run shuttles between King Shaka airport and Pietermaritzburg. They have just purchased a minibus taxi for R1 200 000 and decide to set up a monthly sinking fund to purchase an equivalent new minibus in 8 years' time. The following must be taken into account:

- The current minibus will be traded in to reduce the cost of the new one in 8 years' time.
- The depreciation rate on the current minibus is 15% per annum on the reducing balance.
- The inflation rate on new taxis is estimated to be 5,5% per annum.
- An amount of R10 000 will be withdrawn from the fund at the end of each of the 8 years to pay for a service on the minibus.
- The interest on the investment will be 10% per annum compounded monthly.
- The first payment into the sinking fund will be made immediately and the final payment in 6 years' time.

1.1   Write down the depreciated value of the old taxi after 8 years. (2)

1.2   Write down the cost of a new taxi in 8 years' time. (2)

1.3   Show that, correct to 2 decimal places, the effective interest rate equivalent to a nominal rate of 10% per annum compounding monthly is 10,47% per annum. (2)

1.4   Determine the monthly payment required to ensure the sinking fund covers the annual service and the purchase of the new minibus in 8 years' time. (11)

[17]

QUESTION 2

A loan of P rand is taken on 1 January 2021 and repaid over a number of years, with equal monthly instalments starting one month after the granting of the loan, and then a final smaller payment. The interest on the loan is 9,75% per annum, compounded monthly. After one year, the outstanding balance has ***reduced by*** R36 868,69. The interest paid on the loan in the first year is R47 131,31.

2.1   Deduce that the monthly payment is R7 000. (5)

2.2   Show by calculation that the amount of the original loan is R500 000, to the nearest rand. (8)

2.3   What is the date and value of the final payment? (12)

[25]

QUESTION 3

3.1   The second-order linear recurrence relation (or difference equation)

$T_n = 7T_{n-1} + aT_{n-2}$; $T_1 = p$; $T_2 = q$ has an explicit formula:

$T_n = 4^n - 5\left(3^{n-1}\right)$

Determine the values of a, p and q.   (6)

3.2   The number of fish in a lake popular for fishing has been decreasing by 20%
        per year. To avoid the fish stock becoming depleted, the owner adds 1 000
        fish at the end of each season. At present, there are thought to be 100 000
        fish in the lake.

        (a)   Write down a recursive relation for the number of fish, F_n, in the lake
                at a time n years from now.   (4)

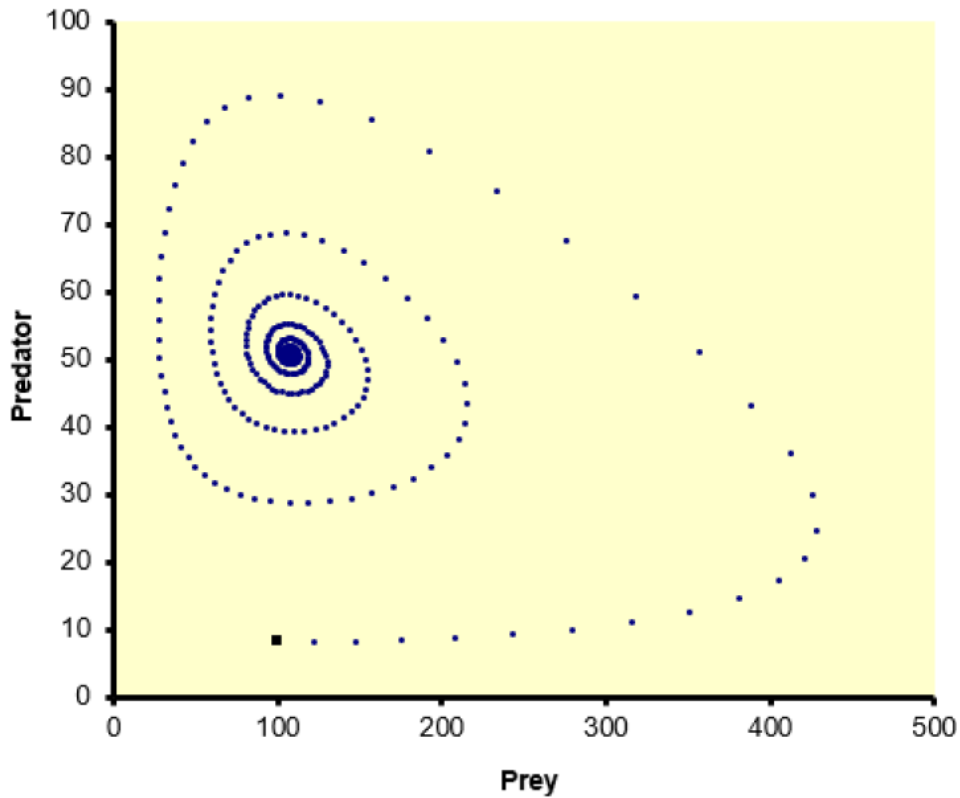   (b)   Determine the equilibrium number of fish.   (3)

        (c)   How many fish must be added each year so that the number of fish
                does not reduce by more than 70% of the current total?   (4)

                                                                                      [17]

QUESTION 4

A typical phase plot of a predator-prey model

4.1   State the initial number of predators and prey.   (2)

4.2   Estimate the equilibrium populations of the predator and the prey.   (2)

4.3   Describe what is happening to the predator population when the prey
      population is at its maximum value.   (2)

4.4   Describe the effect of changing the carrying capacity on the equilibrium point.   (2)

[8]

QUESTION 5

In a certain forest, rabbits form the primary source of food for the foxes but are not hunted by any other species.

The interaction between the foxes and rabbits can be described by the Lotka-Volterra equations in the form shown below. Note that the parameters have been given values and the equations simplified. Assume that the cycle time is one month (or 4 weeks).

$$F_{n+1} = 0,9722F_n + 0,000052893R_nF_n$$
$$R_{n+1} = 2,6R_n - 0,001778R_n^2 - 0,03R_nF_n$$

Given that the rabbits have 6 bunnies per litter and one litter every 6 weeks, and that half the rabbit population is female.

5.1 Prove, by calculation, that:

    (a) The survival rate of the newborn rabbits is 80%   (6)

    (b) The successful attack rate of the foxes on the rabbits is 3%   (3)

    (c) The lifespan of the foxes is 3 years   (3)

    (d) The carrying capacity of the environment is 900   (3)

5.2 Calculate the equilibrium numbers of foxes and rabbits respectively.   (7)

[22]

QUESTION 6

6.1   I invest a lump sum in an account which pays compound interest at the rate
        of 1% per month for the first 3 years, 3% per quarter for the next 4 years and
        6% every 6 months for the following 5 years.

        Determine the effective per annum interest rate over the 12 years.   (6)

6.2   A colony of ants is increasing exponentially with each passing day. Initially
        there are 10 000. After one day, there are 10 100. By what percentage is the
        number of ants increasing each year?   (5)

                                                                                    [11]

                                                Total for Module 3: 100 marks

MODULE 4　　　　MATRICES AND GRAPH THEORY

QUESTION 1

Given A = $\begin{pmatrix} 1 & 4 & -1 & 3 \\ 5 & 2 & -1 & -2 \end{pmatrix}$

1.1　Write down the single matrix created when matrix A is reflected in the line　
　　　$y = \sqrt{3}x$　　(6)

1.2　(a)　Give the matrix of the image (A') of A after a shear by a factor of –3,
　　　　　with the y-axis the invariant line.　　(6)

　　　(b)　Determine the ratio of the area of the shape enclosed by the points in
　　　　　matrix A to the area of the image, A'　　(2)

　　　　　　[14]

QUESTION 2

Given,

$$M = \begin{pmatrix} 7 & 3 & 1 \\ 0 & 2 & 3 \\ 2 & 8 & 0 \end{pmatrix} \qquad N = \begin{pmatrix} x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & x \end{pmatrix}$$

2.1　The *trace* of matrix M, written tr(M), is the sum of the leading diagonal elements
　　　of M.

　　　Solve x, if tr(M) = tr(N)　　(3)

2.2　Create matrix K = M – N　　(2)

2.3　Solve for x if $x \in \mathbb{R}$ and the determinant of matrix K is zero.　　(8)

　　　　　　[13]

QUESTION 3

A software engineer is programming an analog clock face to be displayed on a digital screen. To show the correct time on the screen the end point of each hand must be placed correctly on the coordinate plane. The origin is at the centre of the clock. (Each digital hand is then just a line from the origin to the end point). At 12:00, the end point of the minute hand is set at 6 units on the vertical axis and the end point of the hour hand at 4 units on the vertical axis.

3.1   Calculate the angle moved by the second-hand between each second.   (2)

3.2   Calculate the position, on the digital screen, of the end point of the

(a)   minute-hand at 12:12   (8)

(b)   hour-hand at 12:12   (4)

**[14]**

QUESTION 4
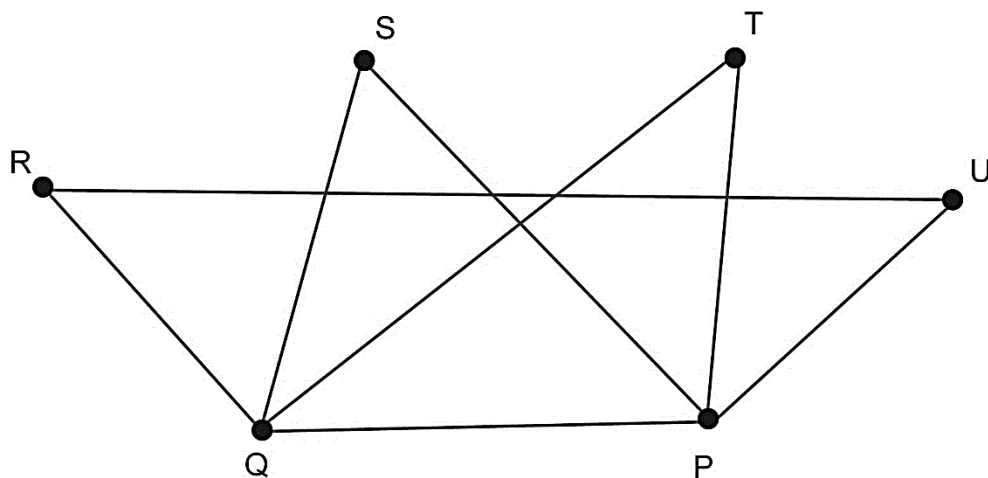
4.1   State the neighbourhood of Q.   (2)

4.2   Draw the complement of the graph. Label the vertices carefully.   (5)

4.3   State the conditions for an Eulerian Circuit to exist.   (2)

4.4   Name one Eulerian Circuit, starting at Q. List the vertices visited in order.   (5)

[14]

QUESTION 5

The royal gardens are designed in an Octagon. Inside the octagon are small concrete paths and walkways with irrigation pipes supplying water to the fountains running under each walkway. Each vertex is a fountain. The weighting of the edges is the walkway distance between each fountain.
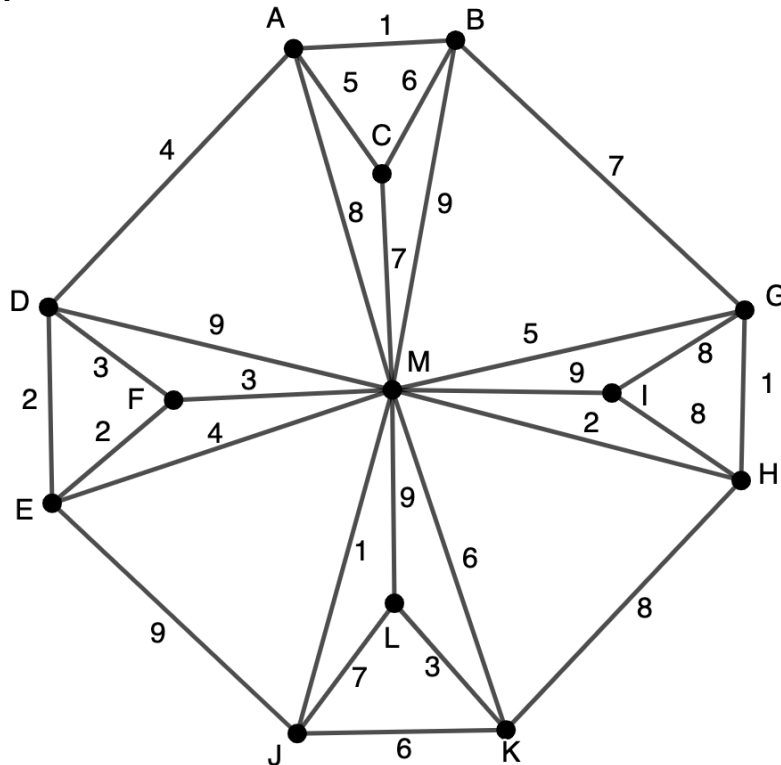
The grounds keeper provides the following list of irrigation piping.
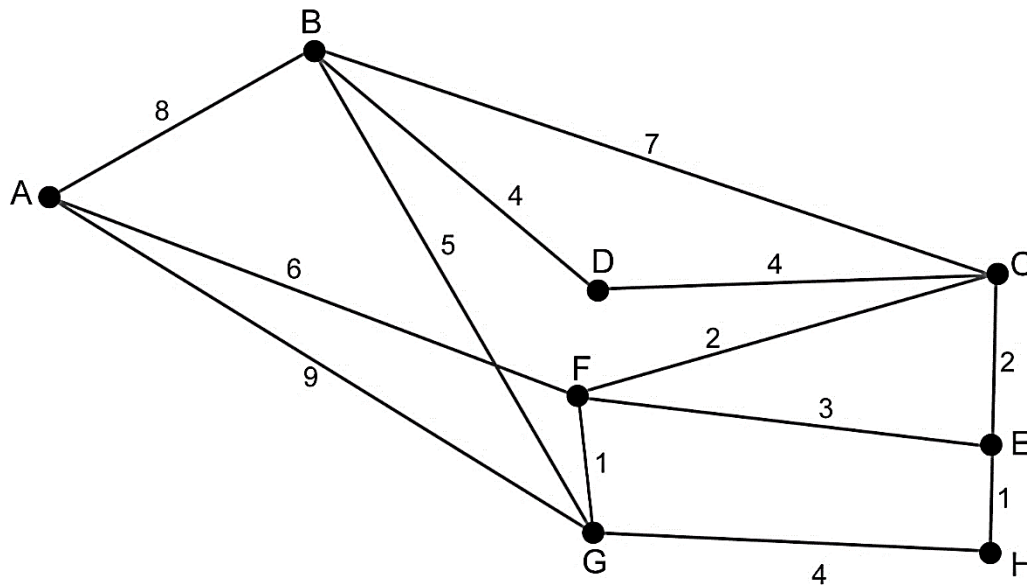
Starting at A:
A–B = 1
M–J = 1
G–H = 1
M–H = 2
E–F = 2
E–D = 2
F–M = 3
L–K = 3
A–D = 4
A–C = 5
M–K = 6 and I–H = 8, for a total distance of 38.

5.1   What is being created, following the edges given above?   (2)

5.2   Which algorithm has been used?   (2)

5.3   Use an alternative algorithm to create a similar structure starting at A.   (8)

5.4   The structure created in Question 5.3 is **not unique**. Which edge(s) should be removed to ensure that it is unique?   (3)

[15]

QUESTION 6

An amateur athlete is in lockdown in her housing complex. She has mapped all the possible running paths, and their times, in the graph below.

The athlete would like to determine a range of possible good running routes. Calculate the range of routes for the above graph, starting at A, by:

6.1   Calculating the lower bound, leaving vertex F out.   (6)

6.2   Determine an upper bound for the athlete's route. Use the Nearest Neighbour Algorithm and clearly record the order in which edges are selected.   (8)

6.3   Use inspection to determine a 'good route' for the athlete to choose.   (3)

[17]

QUESTION 7

If $A = \begin{pmatrix} 7 & 10 \\ 1 & -2 \end{pmatrix}$, $D = \begin{pmatrix} 8 & 0 \\ 0 & -3 \end{pmatrix}$ and $P = \begin{pmatrix} 10 & 1 \\ 1 & -1 \end{pmatrix}$

7.1   Calculate P^{-1}.   (3)

7.2   Show that $A = PDP^{-1}$.   (4)

7.3   Hence, find A^5, showing all matrix calculations.   (6)

[13]

Total for Module 4: 100 marks