GRADE 12 EXAMINATION
NOVEMBER 2019

ADVANCED PROGRAMME MATHEMATICS: PAPER I
MODULE 1: CALCULUS AND ALGEBRA

MARKING GUIDELINES

Time: 2 hours | 200 marks

These marking guidelines are prepared for use by examiners and sub-examiners, all of whom are required to attend a standardisation meeting to ensure that the guidelines are consistently interpreted and applied in the marking of candidates' scripts.

The IEB will not enter into any discussions or correspondence about any marking guidelines. It is acknowledged that there may be different views about some matters of emphasis or detail in the guidelines. It is also recognised that, without the benefit of attendance at a standardisation meeting, there may be different interpretations of the application of the marking guidelines.

QUESTION 1

1.1 Solve for $x \in \mathbb{R}$ without using a calculator and showing all working:

(a) Solve $\left|x^2 - 12\right| = x$

$\therefore x^2 - 12 = x \ \ or \ \ x^2 - 12 = -x$

$\therefore x^2 - x - 12 = 0 \ \ or \ \ x^2 + x - 12 = 0$

$\therefore (x-4)(x+3) = 0 \ \ or \ \ (x+4)(x-3) = 0$

$\therefore x = 4 \ \ or \ \ -3 \ \ or \ \ -4 \ \ or \ \ 3$

$A \ check \ reveals \ x = 4 \ \ or \ \ 3$

(b) $e^x + 12e^{-x} = 8$

$\therefore e^{2x} - 8e^x + 12 = 0$

$\therefore \left(e^x - 2\right)\left(e^x - 6\right) = 0$

$\therefore e^x = 2 \ or \ \ e^x = 6$

$\therefore x = \ln 2 \ \ or \ \ x = \ln 6$

$\therefore x = 0{,}693 \ \ or \ 1{,}792$

1.2 If $z = a + bi$ and $z^2 = 23 - 6z$ then find all possible values of a and b.

$(a+bi)^2 = 23 - 6(a+bi)$

$\therefore a^2 + 2abi + b^2i^2 = 23 - 6a - 6bi$

$\therefore \left(a^2 - b^2\right) + (2ab)i = (23 - 6a) - (6b)i$

$\therefore 2ab = -6b$

$\therefore a = -3 \ \ or \ \ b = 0$

$also \left(a^2 - b^2\right) = 23 - 6a$

$so, if \ a = -3 \ then \ 9 - b^2 = 42$

$\therefore b^2 = -32 \ (not \ possible \ since \ b \ is \ real)$

$\therefore b = 0$

$\therefore a^2 = 23 - 6a$

$\therefore a = -3 + 4\sqrt{2}$

$\therefore a = 2{,}66 \ \ or \ \ -8{,}66 \ \ and \ \ b = 0$

1.3 Solve $f(x)=x^4+x^3-2x^2+2x+4=0$ in $\mathbb{C}$ if it is given that $f(1-i)=0$

if $1-i$ *is a root then so is* $1+i$

so $(x-(1-i))(x-(1+i))$ *is a factor*

so $(x-1)^2-i^2$ *is a factor*

so (x^2-2x+2) *is a factor*

by inspection:

$x^4+x^3-2x^2+2x+4=(x^2-2x+2)(x^2+3x+2)=(x^2-2x+2)(x+1)(x+2)$

$\therefore x=1-i$ *or* $1+i$ *or* -1 *or* -2

QUESTION 2

Use Mathematical Induction to prove that:

$$\sum_{i=1}^{n}2^i=2^{n+1}-2$$

we wish to prove that: $2+2^2+.....+2^n=2^{n+1}-2$

first, let's consider if $n=1$

$LHS=2$ *and* $RHS=2^2-2=2$

so, it is true for $n=1$

Assume it is true for $n=k$

$\therefore 2+2^2+.....+2^k=2^{k+1}-2\ (*)$

adding the next term to each side gives:

$$\begin{aligned}2+2^2+.....+2^k+2^{k+1}&=2^{k+1}-2+2^{k+1}\\&=2\times2^{k+1}-2\\&=2^{k+2}-2\\&=2^{(k+1)+1}-2\end{aligned}$$

but this is just with $n=k+1$

so we have proved that it is true for $n=k+1$

$\therefore$ *by the principle of mathematical induction the result is true for* $n\in\mathbb{N}$

QUESTION 3

Determine $f'(x)$ by first principles if $f(x)=\sqrt{x+3}$

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{x+3+h}-\sqrt{x+3}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{x+3+h}-\sqrt{x+3}}{h}\times\frac{\sqrt{x+3+h}+\sqrt{x+3}}{\sqrt{x+3+h}+\sqrt{x+3}}$$

$$=\lim_{h\to 0}\frac{h}{h\left(\sqrt{x+3+h}+\sqrt{x+3}\right)}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{x+3+h}+\sqrt{x+3}}$$

$$=\frac{1}{2\sqrt{x+3}}$$

QUESTION 4

4.1 Consider the function: $f(x)=\dfrac{x^2+bx-6}{2x-a}$

Determine the values of *a* and *b* if the function has a vertical asymptote at $x=4$ and an oblique asymptote of $y=\dfrac{1}{2}x+4$

For a vertical asymptote at $x=4$ the denominator must be zero when $x=4$ so $a=8$

Now,

$$f(x)=(2x-8)\left(\frac{1}{2}x+4\right)+rem\ (where\ rem\ is\ a\ constant)$$
$$\therefore f(x)=x^2+4x-32+rem$$
$$so,\ b=4$$

4.2 Determine the values of *a* and *b* if the function $f(x)=\dfrac{x^2+ax+b}{2x-3}$ has a stationary point at $(1;2)$

$$we\ know\ that\ f(1)=2\ \ and\ f'(1)=0$$
$$so\ \frac{x^2+ax+b}{2x-3}=\frac{1+a+b}{-1}=2\ \ or\ \ a+b=-3$$
$$\therefore f'(x)=\frac{(2x+a)(2x-3)-2(x^2+ax+b)}{(2x-3)^2}$$
$$now\ f'(1)=(2+a)(-1)-2(1+a+b)=0$$
$$substituting\ the\ value\ of\ a+b\ gives:$$
$$-2-a-2(1-3)=0$$
$$\therefore a=-2-2(-2)=2\ and\ b=-5$$

QUESTION 5

Consider the function *f* defined as follows:

$$f(x)=\begin{cases} 0.5x+4 & x<-4 \\ 3 & -4\le x<-2 \\ 2 & x=-2 \\ 0.5x^2+1 & -2<x<2 \\ g(x) & x\ge 2 \end{cases}$$

Answer the following questions paying careful attention to the **notation** you use:

5.1 Determine $\lim\limits_{x\to -4} f(x)$ if it exists. If not, explain why.

$\lim\limits_{x\to -4^-} f(x)=2$ *but* $\lim\limits_{x\to -4^+} f(x)=3$

$\therefore \lim\limits_{x\to -4} f(x)$ *d.n.e. since they are unequal*

5.2 Why is *f* discontinuous at $x=-2$?

$\lim\limits_{x\to -2} f(x)=3$ *but* $f(-2)=2$

$\therefore$ *discontinuous since they are unequal*

5.3 What type of discontinuity occurs at $x=-2$?

Removable

5.4 Determine $g(x)$ if $g(x)$ is a **linear function** and f is to be differentiable at $x=2$.

we need $g(2)=\lim\limits_{x\to 2^-} f(x)=3$

but we also need $\lim\limits_{x\to 2^+} g'(x)=\lim\limits_{x\to 2^-} f'(x)=\lim\limits_{x\to 2^-} x=2$

so, $g(x)=2x+c$

but $g(2)=2(2)+c=3$

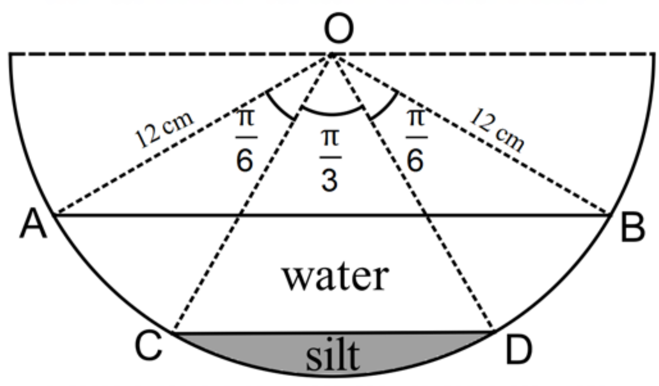
so $c=-1$

$\therefore g(x)=2x-1$

QUESTION 6

Consider the diagram below. It represents the cross-section of a semi-circular gutter with O the centre of the semi-circle. There is silt at the bottom of the gutter. The surface area of the silt CD is parallel to the surface area of the water AB. Angles in radians are as shown. If the radius of the gutter is 12 cm and the gutter is 2 m long then determine the volume of water in the gutter to the nearest litre.

Remember: 1 cm^3 = 1 ml and 1 litre = 1000 ml.



$$\text{Area of water} = \text{minor segment } AB - \text{minor segment } CD$$

$$\text{Now minor segment } AB = \text{sector } AOB - \Delta AOB$$

$$= \frac{1}{2}12^2\frac{2\pi}{3} - \frac{1}{2}12^2\sin\frac{2\pi}{3}$$

$$= 48\pi - \frac{72\sqrt{3}}{2}$$

$$\text{and minor segment } CD = \text{sector } OCD - \Delta OCD$$

$$= \frac{1}{2}12^2\frac{\pi}{3} - \frac{1}{2}12^2\sin\frac{\pi}{3}$$

$$= \frac{72\pi}{3} - \frac{72\sqrt{3}}{2}$$

$$\text{So, area of water} = 48\pi - \frac{72\sqrt{3}}{2} - \left(\frac{72\pi}{3} - \frac{72\sqrt{3}}{2}\right)$$

$$= 24\pi \text{ cm}^2$$

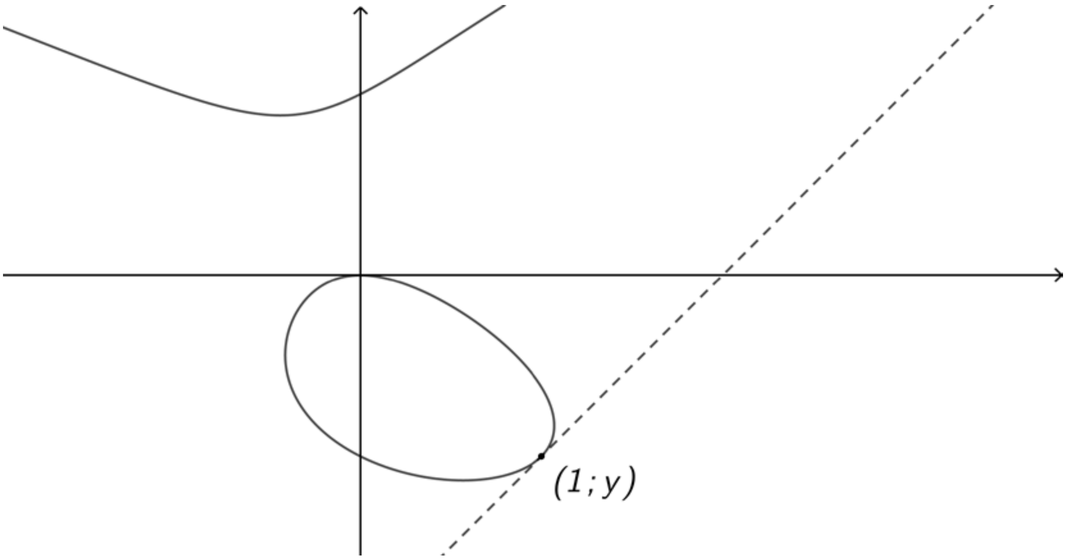
$$\text{So, volume} = 200 \times 24\pi \text{ cm}^3$$

$$= 15079{,}6 \text{ cm}^3$$

$$= 15 \text{ litres to the nearest litre}$$

QUESTION 7

Below is the graph of the implicitly defined relationship: $y^3 - xy = y + x^2$.



Find the equation of the tangent (indicated with a dotted line) if it is known that the x-coordinate of the point of contact is 1.

$y^3 - xy = y + x^2$

$When\ x = 1, y^3 - y = y + 1$

$\therefore y^3 - 2y - 1 = 0$

$\therefore y = -1$

$so, the\ pt.\ of\ contact\ is\ (1;-1)$

$\therefore 3y^2 \frac{dy}{dx} - \left(y + x\frac{dy}{dx} \right) = \frac{dy}{dx} + 2x$

$\therefore 3y^2 \frac{dy}{dx} - y - x\frac{dy}{dx} = \frac{dy}{dx} + 2x$

$\therefore \frac{dy}{dx}\left(3y^2 - x - 1\right) = y + 2x$

$\therefore \frac{dy}{dx} = \frac{y + 2x}{3y^2 - x - 1}$

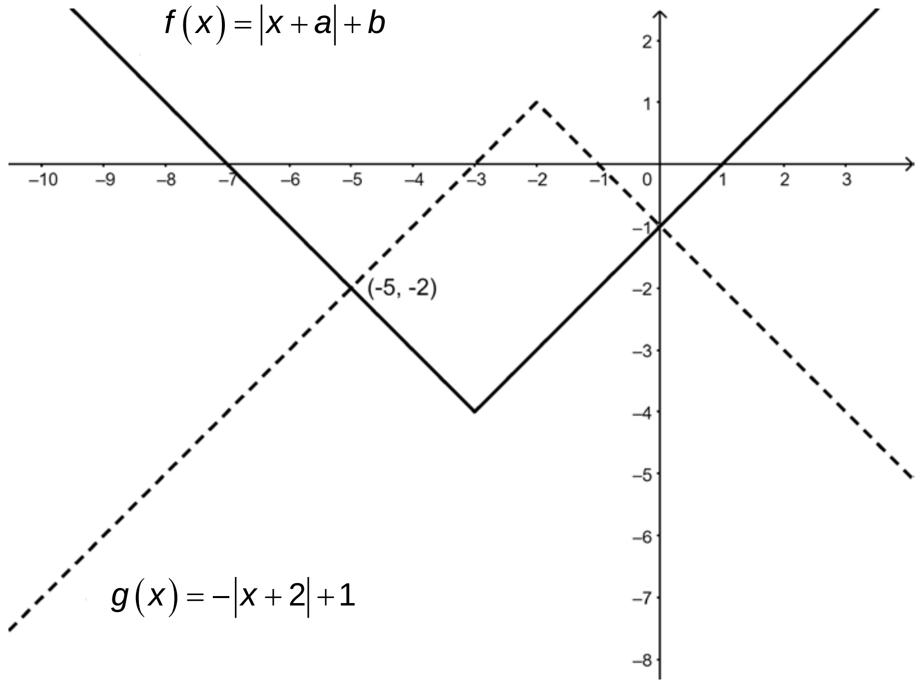
$\therefore \frac{dy}{dx} = \frac{-1 + 2}{3 - 1 - 1} = 1$

$\therefore y - (-1) = 1(x - 1)$

$\therefore y = x - 2$

QUESTION 8

8.1 Consider the functions $f(x) = |x + a| + b$ and $g(x) = -|x + 2| + 1$



(a) Determine the values of *a* and *b*

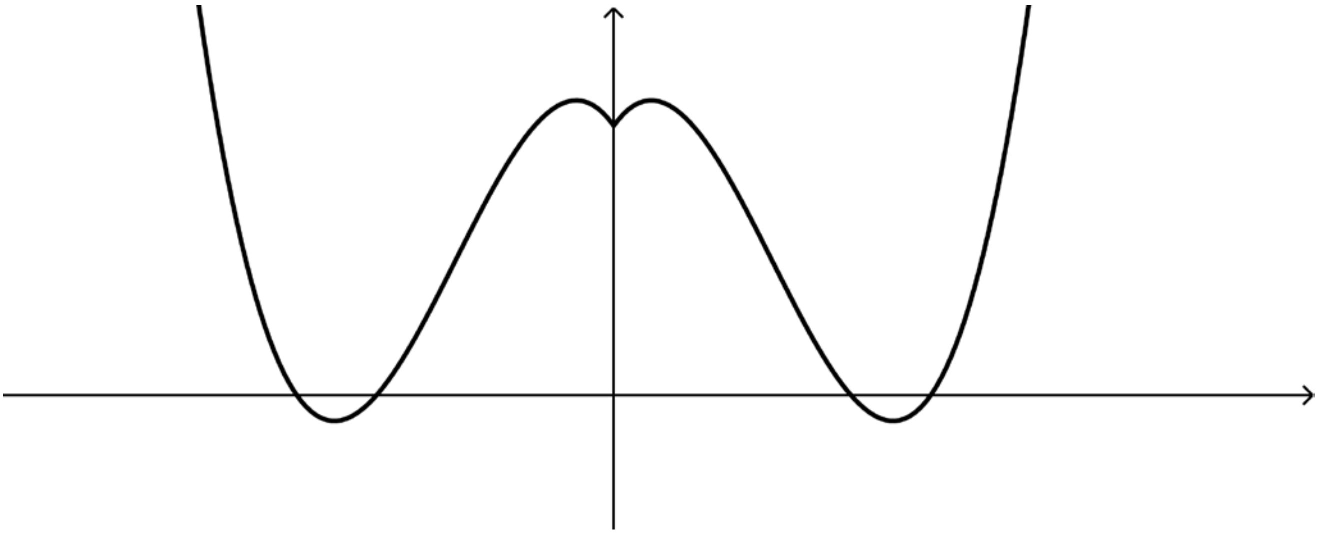
$a = 3$ *and* $b = -4$

(b) Hence or otherwise solve:

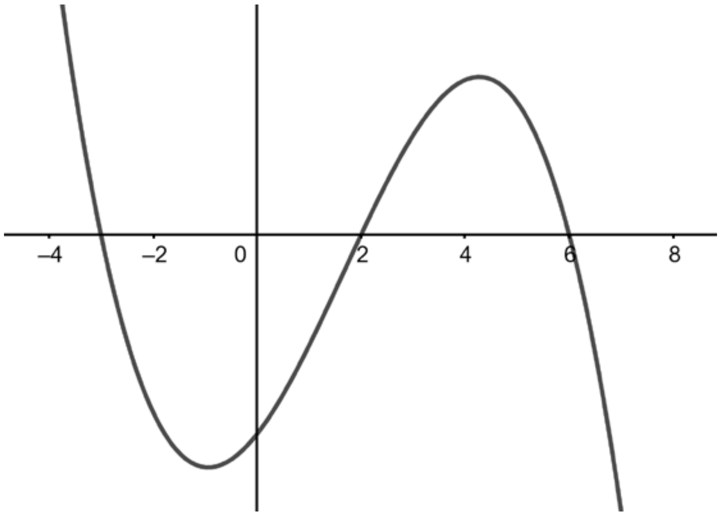
$|x + 3| + |x + 2| \leq 5.$

$$|x+3| + |x+2| \leq 5$$
$$\therefore |x+2| \leq 5 - |x+3|$$
$$\therefore -|x+2| \geq -5 + |x+3|$$
$$\therefore -|x+2| + 1 \geq |x+3| - 4$$
$$\therefore -5 \leq x \leq 0$$

8.2 Given the graph $y = f(x)$, draw on your own set of axes in your Answer Book a rough sketch of $y = f(|x|)$.



8.3 Consider the function, f drawn below.



Given that $\int_0^2 f(x)\,dx = -38,7$ *and* $\int_2^6 f(x)\,dx = 74,7$ determine:

(a) $\int_0^6 f(x)\,dx = -38,7 + 74,7 = 36$

(b) $\int_0^6 |f(x)|\,dx = 38,7 + 74,7 = 113,4$

QUESTION 9

Consider the function $f(x) = x\ln(x) - \sqrt{x^2+4}, \ x > 0$

9.1 Given that *f* is continuous at every value in its domain, justify why *f* has at least one root on the interval $x \in [1;5]$

$f(1) = -\sqrt{5}$

$f(5) = 2,6$

since $f(1) < 0$ *and* $f(5) > 0$

and f *is continuous on* $[1;5]$

f *must cross the* $x-$*axis at least once*

on $[1;5]$ *so there is at least one root on* $[1;5]$

9.2 Use Newton-Raphson iteration to find this root. You should:

- use an initial guess of 1
- show the iterative formula you use
- show your first two approximations
- give your answer to 5 d.p.

$$f(x) = x\ln(x) - \sqrt{x^2+4}$$

$$\therefore f'(x) = \ln(x) + x\left(\frac{1}{x}\right) - \frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}}(2x)$$

$$\therefore f'(x) = \ln(x) + 1 - x\left(x^2+4\right)^{-\frac{1}{2}}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n\ln(x_n) - \sqrt{x_n^{\,2}+4}}{\ln(x_n) + 1 - x_n\left(x_n^{\,2}+4\right)^{-\frac{1}{2}}}$$

$x_0 = 1$

$x_1 = 5,045085...$

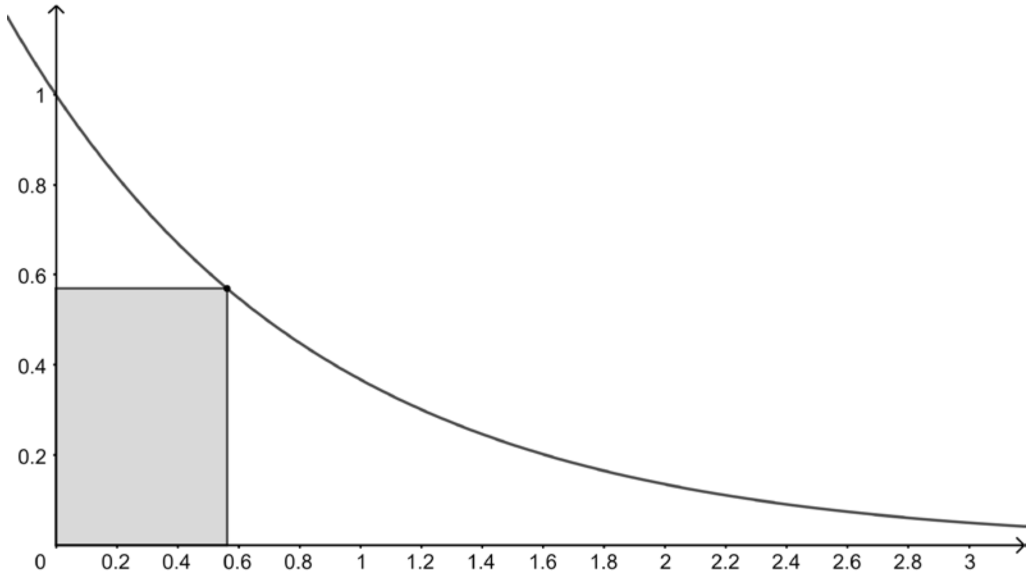
$x_2 = 3,423819...$

$x \approx 3,23903 \ (to\ 5\ d.p.)$

QUESTION 10

Consider the diagram below where rectangles are being formed in the first quadrant. The bottom left corner is on the origin while the top right corner is on the curve $y = e^{-x}$

Find, to 3 decimal places, the maximum area which can be achieved in this way.



$A = xe^{-x}$

$\therefore \dfrac{dA}{dx} = e^{-x} - xe^{-x} = 0$

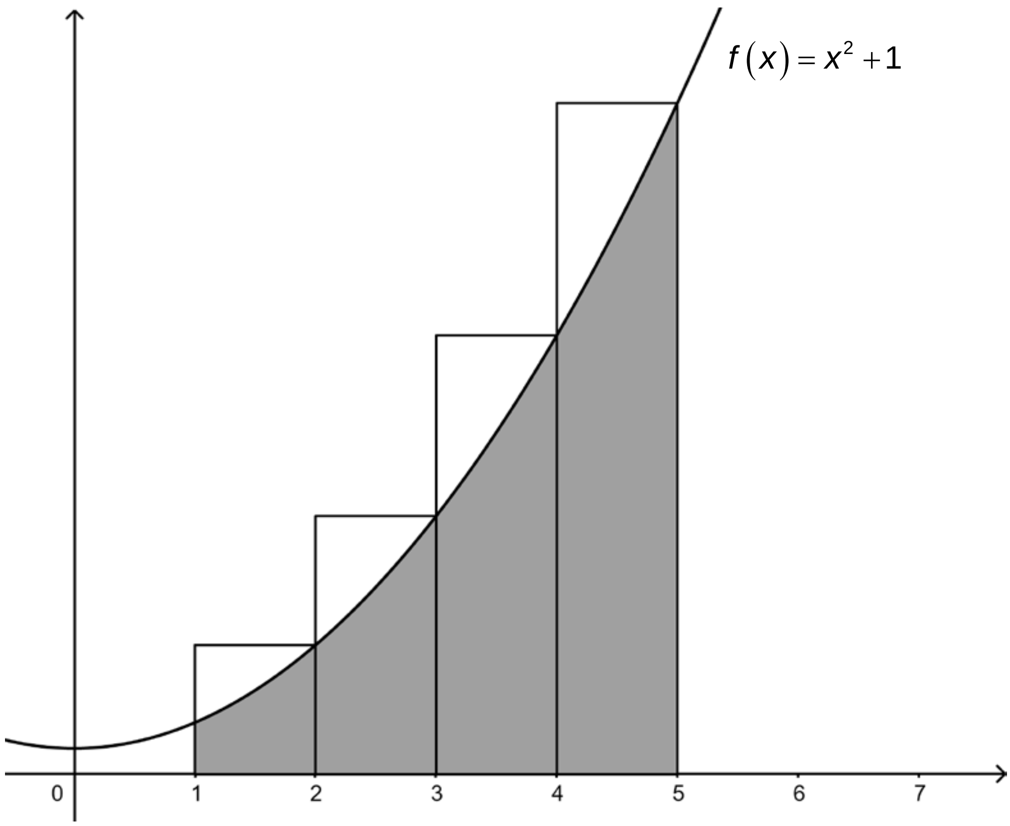
$\therefore e^{-x}\left(1 - x\right) = 0$

$\therefore x = 1$

$\therefore A_{\max} = e^{-1} = 0{,}368\ units^2$

QUESTION 11

11.1 Robyn is using rectangles to estimate the shaded area. Calculate her percentage error to one decimal place.



$Robyn's\ estimate\ is\ (1 \times 5) + (1 \times 10) + (1 \times 17) + (1 \times 26) = 58\ units^2$

$exact\ answer = \int_1^5 x^2 + 1\,dx = \frac{136}{3}\ \ or\ 45\frac{1}{3}\ units^2$

$percentage\ error\ is\ \frac{58}{45\frac{1}{3}} \times 100 = 27{,}9\%$

11.2 (a) Resolve $\frac{3x^2+11x-5}{x^3+3x^2-4}$ into partial fractions.

$$\frac{3x^2+11x-5}{x^3+3x^2-4}=\frac{3x^2+11x-5}{(x-1)(x^2+4x+4)}$$
$$=\frac{3x^2+11x-5}{(x-1)(x+2)^2}$$
$$=\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$$
$$A=1\ by\ cover\ up\ method$$

$$so\ 1(x^2+4x+4)+B(x-1)(x+2)+C(x-1)=3x^2+11x-5 \quad \text{adding}$$
$$x^2+4x+4+Bx^2+Bx-2B+Cx-C=3x^2+11x-5$$
$$=(1+B)x^2+(4+B+C)x+(4-2B-C)$$
$$so, B=2\ and\ C=5$$
$$=\frac{1}{x-1}+\frac{2}{x+2}+\frac{5}{(x+2)^2}$$

(b) Hence, or otherwise, determine $\int\frac{3x^2+11x-5}{x^3+3x^2-4}\,dx$

$$\int\frac{3x^2+11x-5}{x^3+3x^2-4}\,dx$$
$$=\int\frac{1}{x-1}+\frac{2}{x+2}+\frac{5}{(x+2)^2}\,dx$$
$$=\ln|x-1|+2\ln|x+2|-\frac{5}{(x+2)}+c$$

11.3 Determine $\int xe^{2x}\,dx$

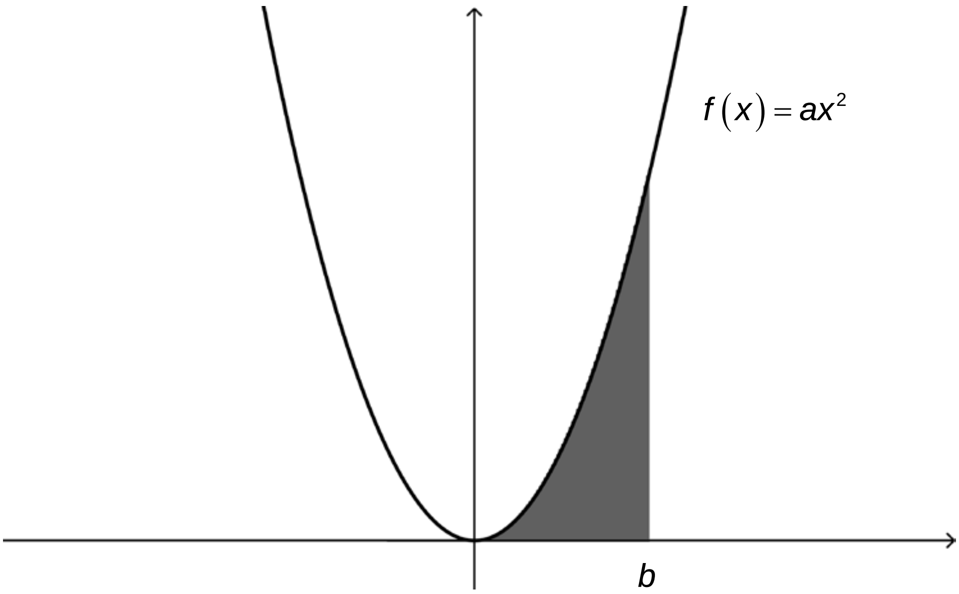
$$let\ f(x)=x\ then\ f'(x)=1$$
$$let\ g'(x)=e^{2x}\ then\ g(x)=\frac{1}{2}e^{2x}$$
$$then\int xe^{2x}\,dx=\frac{1}{2}xe^{2x}-\frac{1}{2}\int e^{2x}\,dx$$
$$=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$$

11.4 Determine $\int \operatorname{cosec}^2 x\cot^2 x\,dx$

$$=-\frac{\cot^3 x}{3}+c$$

QUESTION 12

Consider the diagram below.



The area bounded by the function *f*, the *x*-axis, the lines $x = 0$ and $x = b$ is $\frac{160}{3}$ *units*^{2}.

When this area is rotated around the *x*-axis the resulting volume is 1280π *units*^{3}.

Determine the values of *a* and *b*.

$$\int_0^b ax^2\,dx = \frac{160}{3}$$

$$\therefore \left[\frac{ax^3}{3}\right]_0^b = \frac{160}{3}$$

$$\therefore \frac{ab^3}{3} - 0 = \frac{160}{3}$$

$$\therefore ab^3 = 160 \quad (1)$$

$$\pi\int_0^b \left(ax^2\right)^2\,dx = 1280\pi$$

$$\therefore \left[\frac{a^2x^5}{5}\right]_0^b = 1280$$

$$\therefore \frac{a^2b^5}{5} = 1280$$

$$\therefore a^2b^5 = 6400 \quad (2)$$

now, squaring both sides of equation (1) *gives* $a^2b^6 = 25600\ (3)$

dividing equation (3) *by equation* (2) *gives* $b = 4$

substituting this value into (1) *gives* $a = \frac{5}{2}$

Total: 200 marks